Variations on the Cauchy-Schwarz Inequality*

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ABSTRACT

Let V be a unitary space and let A,B,P,Q be linear on V. A. Abian recently posed the question: what are necessary and sufficient conditions that

$$(Av,u)(Bu,v) \leqslant (Pu,u)(Qv,v)$$

holds for all u,v in V? The principal result of this paper describes these conditions.

I. INTRODUCTION

In a recent query [1] A. Abian posed the following interesting question. Let V be an n-dimensional inner product space, and let A,B,P,Q be linear on V. What relations must exist among these operators so that the inequality

$$(Av,u)(Bu,v) \leqslant (Pu,u)(Qv,v) \tag{1}$$

holds for all u and v? He also suggests investigating the same question with the sense of the inequality reversed. If V is Euclidean (i.e., the underlying field is $\mathbb{R}$), the situation can be chaotic. For example, take $A=I$, $B=-I$, P skew-symmetric, Q anything, and (1) holds.

However, if V is a unitary space (over $\mathbb{C}$) and A,B,P,Q are assumed to be nonsingular, then sensible necessary and sufficient conditions for (1) to hold can be obtained. In the sequel we will prove the following principal result of this paper.

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THEOREM. *Let V be a unitary space,* $\dim V \geqslant 3$, *and assume that* A, B, P, Q *are nonsingular. Then* (1) *holds for all u and v if and only if*

(i) $P = \alpha H$, $Q = \beta K$, $\alpha\beta = \varepsilon = \pm 1$, *$H$ and K are definite hermitian and*
(ii) $A^* = \lambda B$, λ *real, so that* (1) *reads*

$$\lambda |(Bu, v)|^2 \leqslant \varepsilon (Hu, u)(Kv, v), \tag{2}$$

and:

(iii) *if* $\varepsilon = 1$, $\lambda > 0$, *then H and K have the same sign* (*i.e., both positive definite or both negative definite*) *and*

$$\lambda_{\max}(P^{-1}AQ^{-1}B) \leqslant 1; \tag{3}$$

or

(iv) *if* $\varepsilon = 1$, $\lambda < 0$, *then H and K have the same sign*; *or*
(v) *if* $\varepsilon = -1$, $\lambda > 0$, *then H and K have opposite signs and*

$$\lambda_{\max}(P^{-1}AQ^{-1}B) \leqslant 1; \tag{4}$$

or

(vi) *if* $\varepsilon = -1$, $\lambda < 0$, *then H and K have opposite signs.*

($\lambda_{\max}$ is the maximum eigenvalue of the indicated operator.)

It should be noted that for $n = 2$ the inequality (2) can hold with both H and K indefinite. Simply take

$$B = I_2, \qquad H = K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \lambda = \varepsilon = -1.$$

II. SOME PRELIMINARIES

The inner product in V induces an inner product in the second tensor space $V \otimes V$ that satisfies [2, p. 48]

$$(x \otimes y, u \otimes v) = (x, u)(y, v). \tag{5}$$

With respect to the inner product (5) the interchange operator $\sigma : V \otimes V \to V \otimes V$,

$$\sigma x \otimes y = y \otimes x,$$

is obviously hermitian and satisfies $\sigma^2 = I$. It is convenient to interpret (1) in terms of mappings on $V \otimes V$. Let

$$\mathfrak{L} = P \otimes Q - (A \otimes B)\sigma, \tag{6}$$

and note that

$$(\mathfrak{L} u \otimes v, u \otimes v) = (Pu,u)(Qv,v) - (Av,u)(Bu,v). \tag{7}$$

Let $\mathfrak{D}$ denote the set of decomposable elements in $V \otimes V$, i.e., elements of the form $u \otimes v$, u, v in V. Then we see from (6) and (7) that (1) holds if and only if

$$(\mathfrak{L} z, z) \geqslant 0 \qquad \text{for all} \quad z \in \mathfrak{D}. \tag{8}$$

We remark that the condition (8) is not equivalent to $\mathfrak{L} \geqslant 0$ (i.e., $\mathfrak{L}$ positive semidefinite). For let $S_1 = I + \sigma$, $S_2 = I - \sigma$ on a space of dimension 2 and compute that

$$(S_1 - S_2 v_1 \otimes v_2, v_1 \otimes v_2) = \operatorname{per} A - \det A,$$

where A is the 2×2 matrix

$$A = \begin{bmatrix} (v_1, v_1) & (v_1, v_2) \\ (v_2, v_1) & (v_2, v_2) \end{bmatrix}.$$

An inequality of I. Schur [5] states that $\operatorname{per} A \geqslant \det A$. However,

$$(S_1 - S_2) S_2 v_1 \otimes v_2 = -2 S_2 v_1 \otimes v_2.$$

Lemma 1. *If $\mathfrak{L}$ is any linear operator on $V \otimes V$, then $\mathfrak{L} = 0$ if and only if $(\mathfrak{L} z, z) = 0$ for all $z \in \mathfrak{D}$.*

Proof. In

$$(\mathfrak{L} v_1 \otimes v_2, v_1 \otimes v_2) = 0$$

first replace v_2 by $v_2 + x_2$ and then by $v_2 + ix_2$ to conclude that

$$(\mathfrak{L} v_1 \otimes v_2, v_1 \otimes x_2) = 0 \tag{9}$$

and

$$(\mathfrak{L}v_1 \otimes x_2, v_1 \otimes v_2) = 0 \tag{10}$$

for all v_1, v_2, x_2 in V. Then in (9) replace v_1 by $v_1 + y_1$ and then by $v_1 + iy_1$, and use both (9) and (10) to conclude that

$$(\mathfrak{L}v_1 \otimes v_2, y_1 \otimes x_2) = 0 \tag{11}$$

for all v_1, v_2, y_1, x_2 in V. Since arbitrary pairs of decomposable orthonormal basis elements in $V \otimes V$ are of the form $v_1 \otimes v_2, y_1 \otimes x_2$, (11) implies $\mathfrak{L} = 0$. ■

It follows immediately from Lemma 1 that if $(\mathfrak{L}z, z) \in \mathbb{R}$ for all $z \in \mathfrak{D}$, then $\mathfrak{L}$ is hermitian. In particular the inequality (1) requires that both $P \otimes Q$ and $A \otimes B\sigma$ be hermitian.

LEMMA 2. (a) *$P \otimes Q$ is hermitian if and only if $P = \alpha H$, $Q = \beta K$, where H and K are hermitian and $|\alpha| = |\beta| = 1$, $\alpha\beta = \varepsilon = \pm 1$.*

(b) *$A \otimes B\sigma$ is hermitian if and only if $A^* = \lambda B$, $B^* = \mu A$, λ and μ are real and $\lambda\mu = 1$.*

Proof. (a) From $P \otimes Q = (P \otimes Q)^* = P^* \otimes Q^*$ we conclude that [2, p. 83] $P^* = \theta P$, $Q^* = \varphi Q$, $\theta\varphi = 1$, and by taking euclidean norms $|\theta| = |\varphi| = 1$. If $\theta = e^{ir}$, then $(e^{ir/2}P)^* = e^{-ir/2}P^* = e^{-ir/2}e^{ir}P = e^{ir/2}P$. Hence $e^{ir/2}P = H$ is hermitian, and we set $\alpha = e^{-ir/2}$. Similarly $Q = \beta K$ and $\alpha\beta H \otimes K = \alpha H \otimes \beta K = P \otimes Q = (P \otimes Q)^* = P^* \otimes Q^* = \overline{\alpha\beta} H \otimes K$; hence $\alpha\beta = \varepsilon$ is real. Since $|\alpha\beta| = 1$, we conclude $\varepsilon = \pm 1$.

(b) We compute that $(A \otimes B\sigma)^* = \sigma^* A^* \otimes B^* = \sigma A^* \otimes B^*$, and hence $A \otimes B\sigma$ is hermitian if and only if $\sigma A^* \otimes B^* = A \otimes B\sigma$, or equivalently,

$$\sigma A^* \otimes B^* \sigma = A \otimes B.$$

But $\sigma A^* \otimes B^* \sigma = B^* \otimes A^*$, and hence $A \otimes B\sigma$ is hermitian if and only if $A^* = \lambda B$, $B^* = \mu A$, $\lambda\mu = 1$. But $A^* = \lambda B$, $A = \bar{\lambda} B^* = \bar{\lambda}\mu A$, so $\bar{\lambda}\mu = 1$, and similarly $B^* = \mu A$, $B = \bar{\mu} A^* = \bar{\mu}\lambda B$, $\bar{\mu}\lambda = 1$. Since $\lambda\mu = 1$, $\bar{\lambda} = 1/\mu = \lambda \in \mathbb{R}$. ■

We see from (6) and Lemma 2 that

$$\mathfrak{L} = \varepsilon H \otimes K - \lambda (B^* \otimes B)\sigma, \qquad \lambda \in \mathbb{R}, \quad \varepsilon = \pm 1, \tag{12}$$

in which H and K are hermitian, and we can summarize these preliminary results as follows.

LEMMA 3. *The inequality* (1) *holds if and only if*

$$(\mathcal{L}z,z) \geqslant 0, \qquad \text{all} \quad z = u \otimes v \in \mathcal{D}, \tag{13}$$

for the operator (12).

Note that (13) is precisely the same as

$$\varepsilon(Hu,u)(Kv,v) \geqslant \lambda|(Bu,v)|^2, \qquad u,v \in V. \tag{14}$$

III. PROOF OF THE THEOREM

The remainder of the paper consists of a somewhat intricate analysis of the inequality (13) for the operator (12) with various sign alternatives for λ and ε.

LEMMA 4. *If* $\varepsilon = 1$, $\lambda > 0$, *then* (1) *holds if and only if* H *and* K *are definite of the same sign and*

$$\lambda_{\max}(P^{-1}AQ^{-1}B) \leqslant 1.$$

Proof. From Lemma 3 the inequality (1) becomes

$$(Hu,u)(Kv,v) \geqslant \lambda|(Bu,v)|^2, \qquad u,v \in V. \tag{15}$$

Clearly if either H or K were indefinite, the left side of (15) could assume negative values. Thus both are definite and clearly of the same sign. By replacing H with $-H$ and K with $-K$ in (15) if necessary, we may assume that $H > 0$, $K > 0$. Then setting $x = H^{1/2}u$, $y = K^{1/2}v$ in (15) produces the equivalent

$$\|x\|^2 \|y\|^2 \geqslant \lambda\left|(K^{-1/2}BH^{-1/2}x,y)\right|^2, \qquad x,y \in V. \tag{16}$$

But (16) holds for all x, y if and only if the maximum eigenvalue of

$$(K^{-1/2}BH^{-1/2})^*(K^{-1/2}BH^{-1/2}) = H^{-1/2}B^*K^{-1}BH^{-1/2} \tag{17}$$

is at most $1/\lambda$. However, (17) has the same eigenvalues as (Lemma 2)

$$H^{-1}B^*K^{-1}B = \frac{1}{\lambda}P^{-1}AQ^{-1}B,$$

and the result follows. ∎

LEMMA 5. *If $\varepsilon = -1$, $\lambda > 0$, then* (1) *holds if and only if H and K are definite of opposite signs and*

$$\lambda_{\max}(P^{-1}AQ^{-1}B) \leqslant 1. \tag{18}$$

Proof. The inequality (14) becomes

$$(-Hu,u)(Kv,v) \geqslant \lambda|(Bu,v)|^2,$$

which is precisely (15) with $-H$ replacing H. We can apply Lemma 4 to conclude that (1) holds if and only if $-H$ and K are definite of the same sign and the maximum eigenvalue of

$$(-H)^{-1}B^*K^{-1}B = -\frac{1}{\lambda}P^{-1}AQ^{-1}B$$

is at most $1/\lambda$ [see (17)] ■

The cases that remain are $\varepsilon = 1$, $\lambda < 0$ and $\varepsilon = -1$, $\lambda < 0$. For the first of these (14) becomes

$$(Hu,u)(Kv,v) \geqslant -|\lambda|\,|(Bu,v)|^2,$$

or

$$\left|(|\lambda|^{1/2}Bu,v)\right|^2 \geqslant (-Hu,u)(Kv,v). \tag{19}$$

For $\varepsilon = -1$, $\lambda < 0$, (14) becomes

$$-(Hu,u)(Kv,v) \geqslant -|\lambda|\,|(Bu,v)|^2$$

or

$$\left|(|\lambda|^{1/2}Bu,v)\right|^2 \geqslant (Hu,u)(Kv,v). \tag{20}$$

Both (19) and (20) are dealt with as follows.

LEMMA 6. *Let $n \geqslant 3$; assume that B, H, K are nonsingular and that H and K are hermitian. Then*

$$|(Bu,v)|^2 \geqslant (Hu,u)(Kv,v), \qquad u,v \in V, \tag{21}$$

if and only if H and K are of opposite signs, i.e., (21) *can hold only trivially.*

Proof. Note that $n \geqslant 3$ is necessary for this result, as the example following the statement of the Theorem in Sec. I shows.

Let $B = \Delta U$ be the polar factorization of B, $\Delta > 0$, U unitary. Set $x = Uu$, $z = \Delta^{1/2}x$, $w = \Delta^{1/2}v$ in (21) to reduce that inequality to the equivalent

$$|(z,w)|^2 \geqslant (\Delta^{-1/2}UHU^*\Delta^{-1/2}z,z)(\Delta^{-1/2}K\Delta^{-1/2}w,w) \tag{22}$$

for all z,w in V. Suppose we can prove the result with $B = I$. Then (22) tells us that the necessary and sufficient condition for (21) to hold is that $\Delta^{-1/2}UHU^*\Delta^{-1/2}$ and $\Delta^{-1/2}K\Delta^{-1/2}$ have opposite signs. But the first of these is conjunctive to H and the second is conjunctive to K, so (22) and the equivalent (21) hold if and only if H and K have opposite signs. Thus it suffices to prove the result for $B = I$, i.e.,

$$|(u,v)|^2 \geqslant (Hu,u)(Kv,v). \tag{23}$$

Case 1: $n \geqslant 5$. If, say, H were indefinite there would exist u_1, u_2 in V such that $(u_1,u_2) = 0$, $\|u_1\| = \|u_2\| = 1$ and $(Hu_1,u_1) > 0$, $(Hu_2,u_2) < 0$. Then (27) implies that if $v \in \langle u_1 \rangle^{\perp}$

$$0 \geqslant (Hu_1,u_1)(Kv,v)$$

and hence $(Kv,v) \leqslant 0$. If $v \in \langle u_2 \rangle^{\perp}$, then similarly $(Kv,v) \geqslant 0$. Thus if $v \in \langle u_1,u_2 \rangle^{\perp}$, $(Kv,v) = 0$. In other words, if $e_1,\ldots,e_{n-2}$ is an orthonormal basis of $\langle u_1,u_2 \rangle^{\perp}$ and $E = \{e_1,\ldots,e_{n-2},u_1,u_2\}$, then the matrix representation C of K with respect to E is 0 in the first $n-2$ rows and columns. Since $2(n-2) \geqslant n+1$ (i.e., $n \geqslant 5$), the Frobenius-König theorem [4, p. 97] implies that K is singular, a contradiction.

Case 2: $n = 4$. Assume again that H is indefinite. There are two essentially different possibilities.

(i) H has 1 positive and 3 negative eigenvalues. Let u_1 correspond to the positive eigenvalue, u_2, u_3, u_4 correspond to the negative eigenvalues, u_1, u_2, u_3, u_4 orthonormal eigenvectors. Then by the same argument used in Case 1,

$$(Ky,y) \leqslant 0 \qquad \text{for} \quad y \in \langle u_2,u_3,u_4 \rangle; \tag{24}$$

$$(Ky,y) \geqslant 0 \qquad \text{for} \quad \begin{cases} y \in \langle u_1,u_3,u_4 \rangle, & (25) \\ y \in \langle u_1,u_2,u_4 \rangle, & (26) \\ y \in \langle u_1,u_2,u_3 \rangle. & (27) \end{cases}$$

From (24), (25), $(Ky,y)=0$ on $\langle u_3,u_4\rangle$; from (24), (26), $(Ky,y)=0$ on $\langle u_2,u_4\rangle$; from (24), (27), $(Ky,y)=0$ on $\langle u_2,u_3\rangle$. If C is the matrix representation of K on the basis u_1,u_2,u_3,u_4, we conclude that the submatrix of C lying in rows and columns 2,3,4 is 0 and hence C is singular. We remark that if H has 1 negative and 3 positive eigenvalues, then replacing H by $-H$ and K by $-K$ in (23) reduces the problem to (i).

(ii) H has 2 positive and 2 negative eigenvalues. Again let u_1,u_2,u_3,u_4 be orthonormal eigenvectors corresponding in order to the 2 positive and 2 negative eigenvalues. Then as before,

$$(Ky,y) \leqslant 0 \qquad \text{for} \quad \begin{cases} y\in\langle u_2,u_3,u_4\rangle, & (28)\\ y\in\langle u_1,u_3,u_4\rangle; & (29)\end{cases}$$

$$(Ky,y) \geqslant 0 \qquad \text{for} \quad \begin{cases} y\in\langle u_1,u_2,u_4\rangle, & (30)\\ y\in\langle u_1,u_2,u_3\rangle. & (31)\end{cases}$$

From (28), (30), $(Ky,y)=0$ on $\langle u_2,u_4\rangle$; from (28), (31), $(Ky,y)=0$ on $\langle u_2,u_3\rangle$; from (29), (30), $(Ky,y)=0$ on $\langle u_1,u_4\rangle$; from (29), (31), $(Ky,y)=0$ on $\langle u_1,u_3\rangle$. These conditions imply that the matrix C defined above is singular, a contradiction.

Case 3. Again, suppose H is indefinite with 1 negative and 2 positive eigenvalues. Let u_1,u_2,u_3 be orthonormal eigenvectors that correspond in order to the 2 positive and 1 negative eigenvalues. Then as in the above arguments,

$$(Ky,y) \leqslant 0 \qquad \text{for} \quad \begin{cases} y\in\langle u_2,u_3\rangle, & (32)\\ y\in\langle u_1,u_3\rangle, & (33)\end{cases}$$

and

$$(Ky,y) \geqslant 0 \qquad \text{for} \quad y\in\langle u_1,u_2\rangle. \tag{34}$$

Then (32) and (34) imply that $(Ku_2,u_2)=0$, while (33) and (34) imply that $(Ku_1,u_1)=0$. Thus the matrix representation of K with respect to the basis u_1,u_2,u_3 has the form

$$C=\begin{bmatrix} 0 & c_{12} & c_{13}\\ \bar{c}_{12} & 0 & c_{23}\\ \bar{c}_{13} & \bar{c}_{23} & c_{33}\end{bmatrix}.$$

But (34) implies that

$$\begin{bmatrix} 0 & c_{12} \\ \bar{c}_{12} & 0 \end{bmatrix}$$

is positive semidefinite, and hence $c_{12}=0$. But then C and K are singular, a contradiction. The case in which H has 1 positive and 2 negative eigenvalues is obtained from the preceding argument by replacing H with $-H$ and K with $-K$ in (23). ∎

The Theorem in Sec. I is now simply a combination of Lemmas 2, 4, 5, 6. We remark that replacing A by $-A$ and P by $-P$ in the inequality

$$(Av,u)(Bu,v) \geqslant (Pu,u)(Qv,v) \tag{35}$$

reduces (35) to (1), and the Theorem can be applied to the operators $-A$, B, $-P$, Q. We omit the obvious details.

IV. FURTHER INVESTIGATIONS

An extended version of (1) can be formulated for determinants. We first state the problem for matrices. Let A, B, P, Q be n-square complex matrices, and for $1 \leqslant m \leqslant n$, let U and V be $n \times m$ complex matrices. Then generalizing (1) from the $m=1$ case we can state:

$$\det U^*AV \det V^*BU \leqslant \det U^*PU \det V^*QV. \tag{36}$$

It is obvious that in (36) we may assume $\det U^*U = \det V^*V = 1$. The invariant formulation of (36) can then be stated in terms of induced mappings on the Grassmannian manifold $\mathfrak{G}_m$ [3, p. 121]:

$$\begin{aligned}&(C_m(A)v_1\wedge\cdots\wedge v_m, u_1\wedge\cdots\wedge u_m)(C_m(B)u_1\wedge\cdots\wedge u_m, v_1\wedge\cdots\wedge v_m)\\ &\leqslant (C_m(P)u_1\wedge\cdots\wedge u_m, u_1\wedge\cdots\wedge u_m)(C_m(Q)v_1\wedge\cdots\wedge v_m, v_1\wedge\cdots\wedge v_m).\end{aligned} \tag{37}$$

Now let $G < S_{2m}$ be the direct product of the symmetric group S_m on $1,\ldots,m$ with the symmetric group S_m' on $m+1,\ldots,2m$, and let $\chi = \varepsilon\varepsilon'$, where ε (ε') is the alternating character of S_m (S_m'). Form the symmetry class of tensors over

V of degree $2m$ [1, p. 85] associated with G and χ as $imS = V_\chi(G)$, where S is the symmetry operator

$$S = \frac{1}{(m!)^2} \sum_{\theta \in G} \chi(\theta) P(\theta). \tag{38}$$

Note that

$$S = S_\varepsilon S_{\varepsilon'}, \tag{39}$$

where

$$S_\varepsilon = \frac{1}{m!} \sum_{\sigma \in S_m} \varepsilon(\theta) P(\theta), \qquad S_{\varepsilon'} = \frac{1}{m!} \sum_{\varphi \in S'_m} \varepsilon(\varphi) P(\varphi).$$

If $u_1, \ldots, u_m$, $v_1, \ldots, v_m$ are any $2m$ vectors in V, then (39) immediately implies that

$$\begin{aligned} Su_1 \otimes \cdots \otimes u_m \otimes v_1 \otimes \cdots \otimes v_m &= S_\varepsilon S_{\varepsilon'} u_1 \otimes \cdots \otimes u_m \otimes v_1 \otimes \cdots \otimes v_m \\ &= (u_1 \wedge \cdots \wedge u_m) \otimes (v_1 \wedge \cdots \wedge v_m). \end{aligned} \tag{40}$$

It is also obvious that $f \colon \times_1^{2m} V \to V_\chi(G)$ defined by

$$f(u_1, \ldots, u_m, v_1, \ldots, v_m) = v_1 \wedge \cdots \wedge v_m \otimes u_1 \wedge \cdots \wedge u_m$$

is symmetric with respect to G and χ, and hence the universal factorization property of $V_\chi(G)$ produces a linear map $\sigma : V_\chi(G) \to V_\chi(G)$ satisfying

$$\sigma(u_1 \wedge \cdots \wedge u_m \otimes v_1 \wedge \cdots \wedge v_m) = v_1 \wedge \cdots \wedge v_m \otimes u_1 \wedge \cdots \wedge u_m. \tag{41}$$

For simplicity we write $u_1 \otimes \cdots \otimes u_m$ as $u^\otimes$, $u_1 \wedge \cdots \wedge u_m$ as $u^\wedge$, etc., so that (37) now reads

$$\left(C_m(A) \otimes C_m(B) \sigma u^\wedge \otimes v^\wedge, u^\wedge \otimes v^\wedge\right) \leqslant \left(C_m(P) \otimes C_m(Q) u^\wedge \otimes v^\wedge, u^\wedge \otimes v^\wedge\right). \tag{42}$$

Setting

$$\mathfrak{L} = C_m(P) \otimes C_m(Q) - C_m(A) \otimes C_m(B) \sigma, \tag{43}$$

the inequality (42) can then be stated simply as

$$(\mathcal{L}z, z) \geqslant 0 \tag{44}$$

for all decomposable $z = u^{\wedge} \otimes v^{\wedge}$ in $V_{\chi}(G)$. By the usual arguments we can assume that $u_1, \ldots, u_m$ are orthonormal in the exterior product $u_1 \wedge \cdots \wedge u_m$, etc. It is possible to show that the hermitian property of $\mathcal{L}$, $C_m(P) \otimes C_m(Q)$, $C_m(A) \otimes C_m(B)\sigma$ must follow from (42) [or (44)]. However, it should be noted that this cannot be derived by directly applying the $m = 1$ case, because not all elements of $\bigwedge^m V$ are in the Grassmannian $\mathcal{G}_m$. The remainder of this investigation depends on proving the counterparts of Lemma 2, (14), and Lemmas 4, 5, 6.

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